FORMATION OF AN INTENSE CHARGED-PARTICLE BEAM IN THE NEIGHBORHOOD OF A CURVED EMITTER

V. A. Syrovoi

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 6, pp. 28–35, 1968

A local study is made of the flow region and the charge-free region for an axisymmetric regular beam (the normal component of the magnetic field is zero at the emitter). The study is made within the context of hydrodynamic theory. The equation of the beam boundary and the beam potential and normal derivative on it are determined. A solution is obtained for Laplace's equation in the neighborhood of the emitter surface and the equation of the zero-potential shaping electrode is derived. The cases of space-charged-limited (ρ-mode), temperature (T-mode), and nonzero-initial-velocity emission are investigated. The emitting surface and the Cauchy conditions on it are assumed to be defined by analytic functions. A similar problem was solved in [1] for emission in the ρ-mode and zero magnetic field. The results of [2–4] are utilized. Note that [5] also dealt with solution of the beam equations in the neighborhood of a curved emitter.

1. Let $x^1 = x^1(z, R)$, $x^2 = x^2(z, R)$ be an orthogonal coordinate system in the z R meridional plane with metric tensor g_{ik}, and let the emitting surface be $x^1 = x_0^1$. Assume that all the physical and geometrical parameters defining the flow are independent of the azimuth coordinate $x^3 = \psi$. This assumption has two consequences. First, for the azimuth component of the magnetic field Maxwell's equations yield that

$$H_{x^3} = n = H_0 R^{-1} = H_0 \sqrt{g^{33}}, \qquad H_0 = \text{const} \tag{1.1}$$

and lead to the following relationships when $x^1 = x_0^1$:

$$n_S' = \varkappa_2 n, \quad n_P' = k_2 n, \quad n_S'' = (2\varkappa_2^2 + k_1 k_2)\, n, \quad m_S' = \varkappa_1 m, \quad m = H_{x^2}. \tag{1.2}$$

Here and below, $\varkappa_1$ and $\varkappa_2$, k_1 and k_2 are the principal curvatures of the surfaces $x^1 = \text{const}$, $x^2 = \text{const}$, evaluated when $x^1 = x_0^1$; $T = \varkappa_1 + \varkappa_2$ is the total curvature of the emitting surface; S and P are the arc lengths along the curvilinear x^1- and x^2-axes the prime denotes differentiation; and the subscript indicates the variable with respect to which the differentiation is performed. The conditions under which the space is Euclidean, quoted in [1], are utilized in (1.2).

Second, the equations of motion have the solution

$$v_3 = \int_{x_0^1}^{x^1} \sqrt{g}\, H^2 dx^1 = \int_{x_0^1}^{x^1} \sqrt{g_{11}}\, m\, dx^1 .$$

Though the particle paths are spatial curves, the beam boundary is a surface of revolution, given by the equation

$$\frac{g_{22}}{g_{11}} \frac{dx^2}{dx^1} = \frac{v_2}{v_1} . \tag{1.3}$$

Let the flow region and the Laplace region be separated by a surface whose generator cuts the emitter at the point $O(x_0^1, x_0^2)$. We call this the starting point, its coordinates in the z, R system being z_0, R_0. We introduce into the zR plane local Cartesian coordinates X, Y, connected with the emitter surface, where X is directed along the normal, and Y along the tangent to this surface at the starting point (ϑ is the angle between the normal to the emitter at point O and the z-axis of revolution):

$$X = (z - z_0)\cos\vartheta + (R - R_0)\sin\vartheta, \; Y = -(z - z_0)\sin\vartheta + (R - R_0)\cos\vartheta .$$

We require the expansions of the functions $x^1 - x_0^1$, $x^2 - x_0^2$ in X and Y. It can be shown that

$$s_0 = [a_0(x_0^2)]^{1/2}(x^1 - x_0^1) = X + k_1 XY - \frac{1}{4}\frac{a_1}{a_0^{3/2}} X^2 - \frac{1}{2}\varkappa_1 Y^2 +$$

$$
\begin{aligned}
&+\left[\frac{1}{6}\left(\frac{a_1^2}{a_0^3}-\frac{a_2}{a_0^2}\right)-\frac{1}{3}k_1^2\right]X^3+\frac{1}{2}\left(k_{1S}'+\varkappa_1 k_1-\frac{a_1}{a_0^{3/2}}k_1\right)X^2Y+\\
&+\left(-\frac{1}{2}\varkappa_{1S}'+k_1^2+\frac{1}{4}\frac{a_1}{a_0^{3/2}}\varkappa_1\right)XY^2-\left(\frac{1}{6}\varkappa_{1P}'+\frac{1}{2}\varkappa_1 k_1\right)Y^3+\\
&+\left(-\frac{1}{8}\frac{a_3}{a_0^{5/2}}+\frac{13}{48}\frac{a_1a_2}{a_0^{7/2}}-\frac{7}{48}\frac{a_1^3}{a_0^{9/2}}-\frac{7}{24}k_1k_{1S}'-\frac{1}{8}\varkappa_1k_1^2+\frac{1}{6}\frac{a_1}{a_0^{3/2}}k_1^2\right)X^4+\\
&+\left[\frac{1}{6}k_{1S}''+\frac{1}{2}k_1\varkappa_{1S}'+\frac{1}{6}\varkappa_1k_{1S}'-k_1^3-\frac{1}{2}\left(\frac{a_2}{a_0^2}-\frac{a_1^2}{a_0^3}\right)k_1-\right.\\
&\left.-\frac{1}{4}\frac{a_1}{a_0^{3/2}}(k_{1S}'+\varkappa_1k_1)\right]X^3Y+\ldots \qquad (1.4)\\
&p_0=[b_0(x_0^2)]^{1/2}(x^2-x_0^2)=Y+\varkappa_1XY-\frac{1}{4}\frac{b_{02}'}{b_0^{3/2}}Y^2-\frac{1}{2}k_1X^2+\\
&+\left(-\frac{1}{2}k_{1P}'+\varkappa_1^2+\frac{1}{4}\frac{b_{02}'}{b_0^{3/2}}k_1\right)X^2Y-\left(\frac{1}{6}k_{1S}'+\frac{1}{2}\varkappa_1k_1\right)X^3+\\
&+\left(-\frac{1}{24}k_{1S}''-\frac{1}{6}\varkappa_1k_{1S}'+\frac{1}{4}k_1k_{1P}'-\frac{1}{8}k_1^3-\frac{1}{2}\varkappa_1^2k_1-\frac{1}{16}\frac{b_{02}'}{b_0^{3/2}}k_1^2\right)X^4+\ldots
\end{aligned}
$$

Here, $a_k(x^2)$ and $b_k(x^2)$ are the coefficients of the expansions of the metric tensor elements in $x^1 - x_0^1$. All the quantities in (1.4) are evaluated at the starting point.

Using the parametric equation of the beam boundary

$$X = X_e(u),\ \ Y = Y_e(u)$$

we can obtain the function mapping the real axis in the plane w = u + iv on the beam boundary in the plane Z = X + iY:

$$Z = X_e(w) + iY_e(w), \qquad (1.5)$$

Let us also write the parametric equations of the boundary in z, R coordinates and introduce some auxiliary notation:

$$
\begin{aligned}
z &= z_e(u) = z_0 + X_e(u)\cos\vartheta - Y_e(u)\sin\vartheta,\ \ \beta(u) = dz_e/du,\\
R &= R_e(u) = R_0 + X_e(u)\sin\vartheta + Y_e(u)\cos\vartheta,\ \ \alpha(u) = dR_e/du. \qquad (1.6)
\end{aligned}
$$

The solution of Laplace's equation in the axisymmetric case is [6]

$$
\begin{aligned}
2\varphi(u,v) = \mathrm{Re}\Bigg\{&\left[\frac{R_e(w)}{R}\right]^{1/2}V(w)+\frac{2}{\pi}\int_0^v\Big[2R_e\boldsymbol{K}(\sigma)F-\\
&-2R_e[\boldsymbol{K}(\sigma)-\boldsymbol{E}(\sigma)]V\frac{\alpha(z_e-z)-\beta(R_e-R)}{(R_e-R)^2+(z_e-z)^2}+\beta\boldsymbol{E}(\sigma)V\Big]\times\\
&\times\frac{d\xi}{[(R_e+R)^2+(z_e-z)^2]^{1/2}}\Bigg\},\qquad \sigma=\left[\frac{(R_e-R)^2+(z_e-z)^2}{(R_e+R)^2+(z_e-z)^2}\right]^{1/2}.
\end{aligned}
$$

Here, $\mathbf{K}(\sigma)$ and $\mathbf{E}(\sigma)$ are the complete elliptic integrals of the first and second kinds; R_e, z_e, V, F, α, and β are functions of $\zeta = u + i\xi$; V and F define the potential and its normal derivative on the surface bounding the flow region;

$$
\begin{aligned}
z &= z_0 + [\mathrm{Re}\,X_e(w) - \mathrm{Im}\,Y_e(w)]\cos\vartheta - [\mathrm{Im}\,X_e(w) + \mathrm{Re}\,Y_e(w)]\sin\vartheta,\\
R &= R_0 + [\mathrm{Im}\,X_e(w) + \mathrm{Re}\,Y_e(w)]\cos\vartheta + [\mathrm{Re}\,X_e(w) - \mathrm{Im}\,Y_e(w)]\sin\vartheta.
\end{aligned}
$$

Our next aim is to find the forms of expressions (1.5) and (1.6) under different emission conditions and to utilize (1.7) for the case in which the point at which the potential is evaluated is close to the beam boundary and close to the starting point.

To obtain the functions V and F, we use the conditions for the space to be Euclidean; F is evaluated by means of the relationship

$$\frac{\partial\varphi}{\partial v}=\frac{\partial\varphi}{\partial s_0}\left(\frac{\partial s_0}{\partial X}\frac{\partial X}{\partial v}+\frac{\partial s_0}{\partial Y}\frac{\partial Y}{\partial v}\right)+\frac{\partial\varphi}{\partial p_0}\left(\frac{\partial p_0}{\partial X}\frac{\partial X}{\partial v}+\frac{\partial p_0}{\partial Y}\frac{\partial Y}{\partial v}\right).$$

The course of the arguments is the same as in [1], and we thus confine ourselves to a brief summary of the results.

2. In the case of emission in the ρ-mode, the solution of (1.3) is

$$p_0 = \tau_1 s_0^{4/3} + \tau_2 s_0^2 + \tau_3 s_0^{7/3} + \tau_4 s_0^{8/3} + \tau_5 s_0^3, \qquad \tau_k = \text{const}.$$

Using (1.4), we get

$$Y = aX^{4/3} + bX^2 + cX^{7/3} + dX^{8/3} + eX^3,$$

$$a = -\frac{3}{4}\left(\frac{2}{9J}\right)^{1/3} n, \; b = \frac{1}{10}\frac{J_P'}{J} - \frac{1}{20}\frac{nh^2}{J}, \; c = \frac{1}{10}\left(\frac{2}{9J}\right)^{1/3}\left(\frac{31}{14}\varkappa_1 - \varkappa_2\right) n,$$

$$d = \left(\frac{2}{9J}\right)^{2/3}\left[\frac{153}{1120} k_2 n^2 - \frac{81}{160} m m_P' + \left(\frac{249}{5600} n^2 + \frac{81}{1400} m^2\right)\frac{J_P'}{J} - \frac{27}{1120}\frac{nh^4}{J}\right], \tag{2.1}$$

$$e = \frac{1}{30} T_P' + \frac{1}{150}(4\varkappa_1 - \varkappa_2)\frac{J_P'}{J} + \left(\frac{461}{8400}\varkappa_1 - \frac{17}{700}\varkappa_2\right)\frac{n^3}{J} + \left(-\frac{91}{2100}\varkappa_1 + \frac{2}{175}\varkappa_2\right)\frac{nm^2}{J}, \qquad h^2 = m^2 + n^2.$$

Here, $J = J(x^2)$ is the emission current density.

The potential on the beam boundary, whose parametric equations are

$$X_e = u, \quad Y_e = au^{4/3} + bu^2 + cu^{7/3} + du^{8/3} + eu^3,$$

is given by

$$2\varphi = V(u) = Au^{4/3} + Bu^2 + Cu^{7/3} + Du^{8/3} + Eu^3 + Gu^{10/3},$$

$$A = \left(\frac{9J}{2}\right)^{2/3}, \qquad B = \frac{1}{10}h^2, \qquad C = \frac{8}{15}\left(\frac{9J}{2}\right)^{2/3} T,$$

$$D = \left(\frac{9J}{2}\right)^{1/3}\left(\frac{9}{1400}\frac{h^4}{J} - \frac{33}{70} n \frac{J_P'}{J}\right), \quad E = \left(-\frac{421}{1400}\varkappa_1 + \frac{13}{175}\varkappa_2\right) n^2 + \frac{13}{175} T m^2, \tag{2.2}$$

$$G = \left(\frac{9J}{2}\right)^{2/3}\left[\frac{83}{225}(\varkappa_1^2 + \varkappa_2^2) + \frac{157}{450}\varkappa_1\varkappa_2 + \frac{4}{45} k_2 \frac{J_P'}{J} - \frac{4}{45}\frac{J_P''}{J} + \frac{43}{450}\frac{J_P'^2}{J^2} - \frac{29}{1260}(k_2 n^2 + m m_P')\frac{n}{J} - \frac{11}{315} nh^2 \frac{J_P'}{J} + \frac{1}{1750}\frac{h^6}{J^2}\right].$$

The mapping w $\to$ Z and the approximate inverse mapping Z $\to$ w are defined by

$$Z = X + iY = w + i(aw^{4/3} + bw^2 + cw^{7/3} + dw^{8/3} + ew^3),$$

$$w = u + iv = Z - iaZ^{4/3} - {}^4/_3\, a^2 Z^{5/3} + i(2a^3 - b)Z^2 + ({}^{260}/_{81}\, a^4 - {}^{10}/_3\, ab - ic)\, Z^{7/3} + [{}^{11}/_3\, ac + i(-{}^{1309}/_{243}\, a^5 + {}^{77}/_9\, a^2 b - d)]\, Z^{8/3} + [-{}^{28}/_3\, a^6 + 20a^2 b - 4ad - 2b^2 + i({}^2/_9\, a^2 c - e)]\, Z^3. \tag{2.3}$$

The normal derivative of the potential on the boundary is

$$2\partial\varphi/\partial v\,|_{v=0} = F(u) = Hu^{2/3} + Ku^{4/3} + Lu^{5/3} + Mu^2 + Nu^{7/3},$$

$$H = \frac{4}{3}\left(\frac{9J}{2}\right)^{1/3} n, \quad K = \left(\frac{9J}{2}\right)^{2/3}\left(\frac{2}{5}\frac{J_P'}{J} + \frac{8}{45}\frac{nh^2}{J}\right), \quad L = \frac{14}{9}\left(\frac{9J}{2}\right)^{1/3} Tn,$$

$$M = -\frac{2}{7} k_2 n^2 + 2 m m_P' - \left(\frac{138}{175} n^2 + \frac{43}{175} m^2\right)\frac{J_P'}{J} + \frac{43}{350}\frac{nh^4}{J}, \tag{2.4}$$

$$N = \left(\frac{9J}{2}\right)^{2/3}\left[\frac{2}{5} T_P' + \frac{2}{15}(4\varkappa_1 + \varkappa_2)\frac{J_P'}{J} + \left(\frac{1}{270}\varkappa_1 + \frac{38}{135}\varkappa_2\right)\frac{n^3}{J} + \left(\frac{401}{945}\varkappa_1 + \frac{121}{945}\varkappa_2\right)\frac{nm^2}{J}\right].$$

The information contained in expressions (2.1)–(2.4) is sufficient for evaluating the coefficients in the equation of the zero equipotential up to and including X^3. However, we need to know the derivatives of $\exp(4i/3 \operatorname{arc\,tg} v/u)$ and $(u^2 + v^2)^{2/3}$, considered as functions of $\xi = u^{1/3}$, up to and including the sixth order. We thus confine ourselves to the quadratic terms. To this accuracy, the solution of Laplace's equation is given by

$$2\varphi(u, v) = A(u^2 + v^2)^{2/3}\cos\frac{4}{3}\operatorname{arc\,tg}\frac{v}{u} + \frac{3}{5} H (u^2 + v^2)^{5/6}\sin\frac{5}{3}\operatorname{arc\,tg}\frac{v}{u} +$$

$$+B(u^2-v^2)+C(u^2+v^2)^{7/6}\cos\frac{7}{3}\operatorname{arc\,tg}\frac{v}{u}-\frac{1}{2}\frac{Av}{R_0}(u^2+v^2)^{2/3}\times$$
$$\times\cos\left(\vartheta-\frac{4}{3}\operatorname{arc\,tg}\frac{v}{u}\right)+\frac{3}{7}\left(K+\frac{1}{2}\frac{A\cos\vartheta}{R_0}\right)(u^2+v^2)^{7/6}\sin\frac{7}{3}\operatorname{arc\,tg}\frac{v}{u}. \tag{2.5}$$

The explicit equation for the zero equipotential in the w plane is

$$v=\alpha u+\beta u^{4/3}+\gamma u^{5/3}+\delta u^2. \tag{2.6}$$

Using (2.3), we can rewrite (2.6) thus in local Cartesian coordinates:

$$Y=\alpha X+\mu X^{4/3}+\nu X^{5/3}+\lambda X^2,$$
$$\alpha=\operatorname{tg}\frac{3\pi}{8},\quad \mu=-\frac{3}{20}\left(\frac{2}{9J}\right)^{1/3}\alpha(1+\alpha^2)^{2/3}n,$$
$$\nu=\left(\frac{2}{9J}\right)^{2/3}(1+\alpha^2)^{1/3}\left[\left(\frac{3}{40}+\frac{3}{4}\alpha+\frac{9}{8}\alpha^2-\frac{57}{100}\alpha^3\right)n^2+\frac{3}{40}(1-\alpha^2)m^2\right], \tag{2.7}$$
$$\lambda=\left(\frac{73}{420}+\frac{43}{120}\alpha-\frac{351}{2800}\alpha^2-\frac{4397}{18000}\alpha^3+\frac{469}{1200}\alpha^4-\frac{593}{1200}\alpha^5\right)\frac{n^3}{J}+$$
$$+\left(\frac{23}{1680}+\frac{43}{2100}\alpha^2+\frac{19}{400}\alpha^4\right)\frac{nm^2}{J}-\alpha(1+\alpha^2)\left(\frac{2}{15}T+\frac{3}{8}\frac{\sin\vartheta}{R_0}\right)+$$
$$+(1+\alpha^2)\left(\frac{8}{35}\frac{J_P'}{J}+\frac{9}{56}\frac{\cos\vartheta}{R_0}\right).$$

3. In the case of emission to the T-mode, we have, for the beam boundary in terms of p_0 and s_0:

$$p_0=\tau_1 s_0^{3/2}+\tau_2 s_0^2+\tau_3 s_0^{5/2}+\tau_4 s_0^3. \tag{3.1}$$

In the X, Y coordinates this equation becomes

$$Y=aX^{3/2}+bX^2+cX^{5/2}+dX^3,$$
$$a=-\frac{\sqrt{2}}{3}\frac{n}{\varepsilon^{1/2}},\qquad b=\frac{1}{6}\left(\frac{\varepsilon_P'}{\varepsilon}+\frac{nJ}{\varepsilon^2}\right),$$
$$c=\frac{\sqrt{2}}{5}\varepsilon^{-1/2}\left[\frac{1}{3}\frac{J_P'}{\varepsilon}-\frac{5}{9}\frac{J\varepsilon_P'}{\varepsilon^2}+\left(\frac{5}{12}\varkappa_1-\frac{1}{4}\varkappa_2-\frac{5}{12}\frac{J^2}{\varepsilon^3}-\frac{1}{4}\frac{h^2}{\varepsilon}\right)n\right],$$
$$d=\frac{1}{30}T_P'+\frac{1}{90}(4\varkappa_1-\varkappa_2)\frac{\varepsilon_P'}{\varepsilon}-\frac{4}{45}\frac{JJ_P'}{\varepsilon^4}+\frac{4}{27}\frac{J^2\varepsilon_P'}{\varepsilon^4}+\left(-\frac{1}{45}\varkappa_1+\frac{1}{30}\varkappa_2+\right.$$
$$\left.+\frac{8}{81}\frac{J^2}{\varepsilon^3}+\frac{4}{45}\frac{h^2}{\varepsilon}\right)\frac{nJ}{\varepsilon^2}+\frac{2}{45}k_2\frac{n^2}{\varepsilon}-\frac{1}{15}\frac{mm_P'}{\varepsilon}+\frac{2}{45}\frac{h^2\varepsilon_P'}{\varepsilon^2}.$$

Here, $\varepsilon=\varepsilon(x^2)$ is the electric field at the emitter. Notice that, in the absence of a magnetic field and with uniform emission conditions, the trajectory close to the emitter is a cubic parabola with the same coefficient $T_P'/30$) as in the case of the ρ-mode emission. In the electrostatic case, the trajectory curvature $\varkappa_t$ is determined solely by the field when $x^1=x_0^1$:

$$\varkappa_t={}^1/_3(\ln\varepsilon)_P'.$$

The functions realizing the mappings w → Z and Z → w are

$$Z=w+i(aw^{3/2}+bw^2+cw^{5/2}+dw^3),$$
$$w=Z-iaZ^{3/2}-({}^3/_2\,a^2+ib)Z^2+[-{}^7/_2\,ab+i({}^{21}/_8\,a^3-c)]Z^{5/2}+[{}^{81}/_{16}\,a^4-$$
$$-4ac-2b^2+i(9a^2b-d)]Z^3. \tag{3.2}$$

We obtain for the functions V(u) and F(u)

$$V(u)=Au+Bu^{3/2}+Cu^2+Du^{5/2}+Eu^3,$$
$$A=2\varepsilon,\quad B=\frac{4\sqrt{2}}{3}\frac{J}{\varepsilon^{1/2}},\quad C=\varepsilon T-\frac{1}{3}\frac{J^2}{\varepsilon^2},$$
$$D=\frac{2\sqrt{2}}{3}\frac{J}{\varepsilon^{1/2}}\left(\frac{11}{10}T+\frac{1}{6}\frac{J^2}{\varepsilon^3}-\frac{n\varepsilon_P'}{J}+\frac{1}{10}\frac{h^2}{\varepsilon}\right),$$
$$E=\varepsilon\left[\frac{1}{3}(2\varkappa_1^2+\varkappa_2^2+2\varkappa_1\varkappa_2)-\frac{1}{5}T\frac{J^2}{\varepsilon^3}+\frac{1}{3}\left(k_2\frac{\varepsilon_P'}{\varepsilon}-\frac{\varepsilon_P''}{\varepsilon}+\frac{\varepsilon_P'^2}{\varepsilon^2}\right)-\right.$$
$$\left.-\frac{8}{81}\frac{J^4}{\varepsilon^6}-\frac{2}{9}\varkappa_1\frac{n^2}{\varepsilon}-\frac{7}{9}\frac{nJ_P'}{\varepsilon^2}+\frac{2}{3}\frac{nJ\varepsilon_P'}{\varepsilon^3}-\frac{4}{45}\frac{h^2J^2}{\varepsilon^4}\right],$$

$$F(u) = Gu^{1/2} + Ku + Lu^{3/2} + Mu^2 ,$$

$$G = \sqrt{2\varepsilon}\, n, \quad K = \frac{4}{3}\left(\varepsilon_p' + \frac{nJ}{\varepsilon}\right),$$

$$L = \sqrt{2\varepsilon}\left[\frac{J_p'}{\varepsilon} - \frac{7}{9}\frac{J\varepsilon_p'}{\varepsilon^2} + \left(\frac{5}{4}T - \frac{7}{12}\frac{J^2}{\varepsilon^3} + \frac{1}{4}\frac{h^2}{\varepsilon}\right)n\right], \tag{3.3}$$

$$M = \frac{4}{5}\varepsilon T_p' + \left(\frac{26}{15}\varkappa_1 + \frac{3}{4}\varkappa_2\right)\varepsilon_p' - \frac{4}{5}\frac{JJ_p'}{\varepsilon^2} + \frac{10}{9}\frac{J^2\varepsilon_p'}{\varepsilon^3} + \frac{22}{15}T\frac{nJ}{\varepsilon} -$$

$$- \frac{4}{15}k_2 n^2 + \frac{2}{5}mm_p' - \left(\frac{14}{15}n^2 + \frac{4}{15}m^2\right)\frac{\varepsilon_p'}{\varepsilon} + \left(\frac{20}{27}\frac{J^2}{\varepsilon^2} + \frac{2}{15}h^2\right)\frac{nJ}{\varepsilon^2}.$$

Retaining the quadratic terms, (1.7) gives

$$2\varphi(u, v) = Au + (u^2 + v^2)^{3/4}\left(B\cos\frac{3}{2}\operatorname{arc\,tg}\frac{v}{u} + \frac{2}{3}G\sin\frac{3}{2}\operatorname{arc\,tg}\frac{v}{u}\right) +$$

$$+ C(u^2 - v^2) + Kuv - \frac{1}{2}\frac{A\sin\vartheta}{R_0}v^2. \tag{3.4}$$

Only the first two terms in the explicit equation for the zero equipotential

$$u = \alpha v^{3/2} + \beta v^2 + \gamma v^{5/2} + \delta v^3. \tag{3.5}$$

can be found from (3.4). The other two coefficients can be found by taking direct account of the smallness of the ratio u/v (see (3.5)) when estimating the integral in (1.7). We find as a result, on recalling (3.2), that

$$X = \mu Y^{3/2} + \nu Y^2 + \lambda Y^{5/2} + \tau Y^3,$$

$$\mu = \frac{2}{3}\frac{J}{\varepsilon^{3/2}}, \qquad \nu = \frac{1}{2}T - \frac{7}{6}\frac{J^2}{\varepsilon^3} - \frac{1}{3}\frac{n^2}{\varepsilon} + \frac{1}{2}\frac{\sin\vartheta}{R_0},$$

$$\lambda = \varepsilon^{-1/2}\left[-\frac{2}{15}T\frac{J}{\varepsilon} + \frac{4}{15}\frac{J_p'}{\varepsilon} - \frac{23}{30}\frac{J\varepsilon_p'}{\varepsilon^2} + \frac{7}{9}\frac{J^3}{\varepsilon^4} - \frac{2}{15}\varkappa_2 n + \frac{16}{45}\frac{nJ^2}{\varepsilon^3} + \right.$$

$$\left. + \left(\frac{23}{90}n^2 + \frac{1}{30}m^2\right)\frac{J}{\varepsilon^2} - \frac{5}{36}\frac{n^3}{\varepsilon} - \left(\frac{1}{6}\frac{J}{\varepsilon} + \frac{2}{15}n\right)\frac{\sin\vartheta}{R_0} - \frac{1}{5}\frac{J}{\varepsilon}\frac{\cos\vartheta}{R_0}\right],$$

$$\tau = \frac{1}{6}T_p' - \frac{79}{360}\varkappa_2\frac{\varepsilon_p'}{\varepsilon} + \frac{2}{9}T\frac{J^2}{\varepsilon^3} - \frac{3}{5}\frac{JJ_p'}{\varepsilon^3} + \frac{11}{9}\frac{J^2\varepsilon_p'}{\varepsilon^4} + \tag{3.6}$$

$$+ \frac{35}{81}\frac{J^4}{\varepsilon^6} + \left(-\frac{1}{18}\varkappa_1 + \frac{1}{90}\varkappa_2\right)\frac{n^2}{\varepsilon} +$$

$$+ \left(\frac{1}{9}\varkappa_1 + \frac{23}{90}\varkappa_2\right)\frac{nJ}{\varepsilon^2} - \frac{1}{30}\frac{nJ_p'}{\varepsilon^2} - \left(\frac{1}{36}\frac{nJ}{\varepsilon} + \frac{1}{54}n^2\right)\frac{\varepsilon_p'}{\varepsilon^2} -$$

$$- \frac{10}{9}\frac{nJ^3}{\varepsilon^5} + \left(\frac{1}{36}n^2 + \frac{1}{18}m^2\right)\frac{J^2}{\varepsilon^4} + \frac{1}{180}\frac{nm^2J}{\varepsilon^3} + \left(-\frac{89}{3240}n^2 + \frac{1}{40}m^2\right)\frac{n^2}{\varepsilon^2} +$$

$$+ \frac{113}{3240}\frac{n^3J}{\varepsilon^3} + \left(-\frac{1}{6}T + \frac{1}{6}\frac{J^2}{\varepsilon^3} - \frac{1}{3}\frac{nJ}{\varepsilon^2} + \frac{7}{18}\frac{n^2}{\varepsilon}\right)\frac{\cos\vartheta}{R_0} +$$

$$+ \left(-\frac{5}{18}\frac{\varepsilon_p'}{\varepsilon} + \frac{1}{6}\frac{J^2}{\varepsilon^3} + \frac{1}{4}\frac{nJ}{\varepsilon^2} + \frac{11}{72}\frac{n^2}{\varepsilon}\right)\frac{\sin\vartheta}{R_0} - \frac{1}{6}\frac{\sin 2\vartheta}{R_0^2}.$$

4. In the case of nonzero initial velocity $v_{x1} = u \neq 0$, the beam boundary is given by

$$p_0 = \tau_1 s_0^2 + \tau_2 s_0^3 + \tau_3 s_0^4.$$

or in terms of X, Y by

$$Y = aX^2 + bX^3 + cX^4,$$

$$a = -\frac{1}{2}\frac{n}{u}, \qquad b = \frac{1}{6}\frac{\varepsilon_p'}{u^2} - \frac{1}{6}\varkappa_2\frac{n}{u} + \frac{1}{3}\frac{n\varepsilon}{u^3},$$

$$c = \frac{1}{24}\frac{\varepsilon}{u^2}T_p' + \left(\frac{1}{8}\varkappa_1 + \frac{1}{24}\varkappa_2\right)\frac{\varepsilon_p'}{u^2} + \frac{1}{24}\frac{J_p'}{u^3} - \frac{5}{24}\frac{\varepsilon\varepsilon_p'}{u^4} - \frac{1}{12}\varkappa_2^2\frac{n}{u} + \tag{4.1}$$

$$+ \left(\frac{1}{6}\varkappa_1 + \frac{1}{4}\varkappa_2\right)\frac{n\varepsilon}{u^3} + \frac{1}{24}k_2\frac{n^2}{u^2} - \frac{1}{12}\frac{mm_p'}{u^2} + \left(\frac{1}{8}\frac{J}{u} - \frac{3}{8}\frac{\varepsilon^2}{u^2} - \frac{1}{8}h^2\right)\frac{n}{u^3}.$$

The mapping $w \rightarrow Z$ and $Z \rightarrow w$ are given by

$$Z = w + i(aw^2 + bw^3 + cw^4),$$

$$w = Z - ia\,Z^2 - (2a^2 + ib)\,Z^3 + [-5\,ab + i(5a^3 - c)]\,Z^4. \tag{4.2}$$

We obtain for the potential and its normal derivative on the beam boundary:

$$
\begin{aligned}
&V(u) = Au + Bu^2 + Cu^3 + Du^4, \qquad A = 2\varepsilon,\ B = \varepsilon T + J/u,\\
&C = \frac{2}{3}\varepsilon(\varkappa_1^2 + \varkappa_2^2 + \varkappa_1\varkappa_2) + \left(\frac{2}{3}T - \frac{1}{3}\frac{\varepsilon}{u^2}\right)\frac{J}{u} + \frac{1}{3}k_2\varepsilon_P' - \frac{1}{3}\varepsilon_P'' - \frac{n\varepsilon_P'}{u},\\
&D = -\frac{1}{12}\varepsilon T_P'' + \frac{1}{12}\varepsilon k_2 T_P' + \left(-\frac{1}{3}\varkappa_{1P}' + \frac{1}{2}\varkappa_1 k_2 + \frac{1}{6}\varkappa_2 k_2\right)\varepsilon_P' +\\
&+ \frac{1}{2}\varepsilon(\varkappa_1^2 + \varkappa_2^2)T + \frac{1}{2}(\varkappa_1^2 + \varkappa_2^2 + \varkappa_1\varkappa_2)\frac{J}{u} - \left(\frac{1}{2}\varkappa_1 + \frac{1}{6}\varkappa_2\right)\varepsilon_P'' +\\
&+ \frac{1}{12}k_2\frac{J_P'}{u} - \frac{1}{3}T\frac{J\varepsilon}{u^3} - \frac{1}{12}\frac{J_P''}{u} + \frac{1}{3}\frac{\varepsilon_P'^2}{u^2} - \frac{1}{12}\frac{J^2}{u^4} + \frac{1}{4}\frac{J\varepsilon^2}{u^5} -\\
&-\left(\frac{3}{2}\varkappa_1 + \frac{5}{6}\varkappa_2\right)\frac{n\varepsilon_P'}{u} - \frac{5}{12}\frac{nJ_P'}{u^2} - \frac{1}{4}\varkappa_1\frac{n^2\varepsilon}{u^2} + \frac{2}{3}\frac{n\varepsilon\varepsilon_P'}{u^3} + \frac{1}{12}\frac{h^2 J}{u^3} - \frac{1}{2}T_P'\frac{n\varepsilon}{u},\\
&F(u) = Eu + Ku^2 + Lu^3, \qquad E = 2(\varepsilon_P' + n\varepsilon/u),\\
&K = \varepsilon T_P' + (3\varkappa_1 + \varkappa_2)\varepsilon_P' + \frac{J_P'}{u} - \frac{\varepsilon\varepsilon_P'}{u^2} + \left(3\varepsilon T + 2\frac{J}{u} - 2\frac{\varepsilon^2}{u^2}\right)\frac{n}{u},\\
&L = \varepsilon\left(\frac{7}{3}\varkappa_1 + \frac{2}{3}\varkappa_2\right)\varkappa_{1P}' + \varepsilon\left(\frac{5}{3}\varkappa_1 + \frac{4}{3}\varkappa_2\right)\varkappa_{2P}' + \left(\frac{2}{3}\frac{J}{u} - \frac{1}{3}\frac{\varepsilon^2}{u^2}\right)T_P' +\\
&+\left(\frac{1}{3}k_{2P}' + \frac{11}{3}\varkappa_1^2 + \frac{2}{3}\varkappa_2^2 + \frac{5}{3}\varkappa_1\varkappa_2\right)\varepsilon_P' + \frac{1}{3}k_2\varepsilon_P'' + \left(\frac{5}{3}\varkappa_1 + \frac{2}{3}\varkappa_2\right)\frac{J_P'}{u} -\\
&-\left(\frac{7}{3}\varkappa_1 + \frac{4}{3}\varkappa_2\right)\frac{\varepsilon\varepsilon_P'}{u^2} - \frac{1}{3}\varepsilon_P''' - \frac{2}{3}\frac{\varepsilon J_P'}{u^3} - \frac{4}{3}\frac{J\varepsilon_P'}{u^3} + \frac{5}{3}\frac{\varepsilon^2\varepsilon_P'}{u^4} +\\
&+\left(4\varkappa_1^2 + \frac{11}{3}\varkappa_2^2 + \frac{13}{3}\varkappa_1\varkappa_2\right)\frac{n\varepsilon}{u} + \frac{n}{u}\left(3\frac{J}{u} - 4\frac{\varepsilon^2}{u^2}\right)T + \frac{n}{u}\left(\varepsilon_P' - \frac{1}{3}\frac{n\varepsilon}{u}\right)k_2 +\\
&+\frac{2}{3}\varepsilon\frac{mm_P'}{u^2} - 2\frac{n\varepsilon_P''}{u} - \frac{n^2\varepsilon_P'}{u^2} + \left(-4\frac{J}{u} + 3\frac{\varepsilon^2}{u^2} + h^2\right)\frac{n\varepsilon}{u^3}.
\end{aligned}
\tag{4.3}
$$

For the potential in the Laplace region, we have

$$
\begin{aligned}
&2\varphi(u, v) = Au + Bu^2 + Cu^3 + Du^4 + (Eu + Ku^2 + Lu^3)v -\\
&-\left(B + \frac{1}{2}\frac{A\sin\vartheta}{R_0}\right)v^2 + \left(-3C - \frac{aA\cos\vartheta}{R_0} - \frac{B\sin\vartheta}{R_0} - \frac{1}{2}\frac{E\cos\vartheta}{R_0} + \frac{1}{2}\frac{A\sin^2\vartheta}{R_0^2}\right)uv^2 +\\
&+\left(-\frac{1}{3}K + \frac{1}{3}\frac{aA\sin\vartheta}{R_0} + \frac{1}{3}\frac{B\cos\vartheta}{R_0} - \frac{1}{6}\frac{E\sin\vartheta}{R_0} + \frac{1}{6}\frac{A\sin 2\vartheta}{R_0^2}\right)v^3 +\\
&+\left(-6D - \frac{3}{2}\frac{bA\cos\vartheta}{R_0} - 2\frac{aB\cos\vartheta}{R_0} - \frac{3}{2}\frac{C\sin\vartheta}{R_0} + \frac{aE\sin\vartheta}{R_0} - \frac{1}{2}\frac{K\cos\vartheta}{R_0} +\right.\\
&\left.+\frac{3}{4}\frac{aA\sin 2\vartheta}{R_0^2} + \frac{B\sin^2\vartheta}{R_0^2} + \frac{1}{4}\frac{E\sin 2\vartheta}{R_0^2} - \frac{1}{2}\frac{A\sin^3\vartheta}{R_0^3}\right)u^2v^2 +\\
&+\left(D + \frac{1}{2}\frac{bA\cos\vartheta}{R_0} + \frac{1}{2}\frac{C\sin\vartheta}{R_0} + \frac{1}{6}\frac{K\cos\vartheta}{R_0} - \frac{1}{3}\frac{aA\sin 2\vartheta}{R_0^2} - \frac{1}{12}\frac{B\sin^2\vartheta}{R_0^2} -\right.\\
&\left.-\frac{1}{4}\frac{B\cos^2\vartheta}{R_0^2} + \frac{1}{24}\frac{E\sin 2\vartheta}{R_0^2} + \frac{1}{24}\frac{A\sin^3\vartheta}{R_0^3} - \frac{1}{8}\frac{A\cos\vartheta\sin 2\vartheta}{R_0^3}\right)v^4 +\\
&+\left(-L + \frac{bA\sin\vartheta}{R_0} + \frac{C\cos\vartheta}{R_0} - \frac{1}{3}\frac{K\sin\vartheta}{R_0} - \frac{aA\sin^2\vartheta}{R_0^2} + \frac{2}{3}\frac{aA\cos^2\vartheta}{R_0^2} +\right.\\
&\left.+\frac{1}{6}\frac{B\sin 2\vartheta}{R_0^2} + \frac{1}{6}\frac{E\sin^2\vartheta}{R_0^2} + \frac{1}{3}\frac{E\cos^2\vartheta}{R_0^2} - \frac{1}{3}\frac{A\sin\vartheta\sin 2\vartheta}{R_0^3}\right)uv^3.
\end{aligned}
\tag{4.4}
$$

4.1. Case of nonzero electric field $\varepsilon \neq 0$. Confining ourselves to the first two terms of the expansion, we have for the zero equipotential in the w-plane:

$$u = \alpha v^2 + \beta v^3. \tag{4.5}$$

Using (4.2), we transform to the variables X, Y:

$$
\begin{aligned}
&X = \mu Y^2 + \nu Y^3, \qquad \mu = \frac{1}{2}\left(T + \frac{J}{\varepsilon u} + \frac{\sin\vartheta}{R_0}\right),\\
&\nu = \frac{1}{6}T_P' - \frac{1}{3}\varkappa_2\frac{\varepsilon_P'}{\varepsilon} + \frac{1}{6}\frac{J_P'}{\varepsilon u} - \frac{1}{2}\frac{J}{u}\frac{\varepsilon_P'}{\varepsilon^2} - \frac{1}{6}\left(\varkappa_2 + \frac{J}{\varepsilon u}\right)\frac{n}{u} -\\
&-\frac{1}{6}\left(T + \frac{J}{\varepsilon u}\right)\frac{\cos\vartheta}{R_0} - \frac{1}{6}\left(2\frac{\varepsilon_P'}{\varepsilon} + \frac{n}{u}\right)\frac{\sin\vartheta}{R_0} - \frac{1}{6}\frac{\sin 2\vartheta}{R_0^2}.
\end{aligned}
\tag{4.6}
$$

The curvature at the origin k_φ of the zero shaping electrode is equal to 2μ, while its derivative $k_\varphi' = 6\nu$.

4.2. Case of zero electric field $\varepsilon = 0$. Now, $A = E = 0$, while the surface $\varphi = 0$ is given by

$$v = \alpha u + \beta u^2 + \gamma u^3. \tag{4.7}$$

In terms of X, Y (4.7) may be rewritten as

$$Y = \alpha X + \nu X^2 + \lambda X^3$$

$$\alpha = 1, \quad \nu = -\frac{2}{3} T + \frac{1}{3} \frac{J_P'}{J} - \frac{1}{3} \frac{n}{u} - \frac{1}{2} \frac{\sin\vartheta}{R_0} + \frac{1}{6} \frac{\cos\vartheta}{R_0},$$

$$\lambda = \frac{1}{9}(\varkappa_1^2 + \varkappa_2^2 + 11\varkappa_1\varkappa_2) - \left(\frac{4}{9} T + \frac{1}{6} k_2\right) \frac{J_P'}{J} + \frac{1}{6} \frac{J_P''}{J} - \frac{1}{18} \frac{J_P'^2}{J^2} + \tag{4.8}$$

$$+ \frac{1}{6} \frac{J}{u^3} + \left(\frac{4}{9} T - \frac{1}{18} \frac{J_P'}{J}\right) \frac{n}{u} + \frac{5}{18} \frac{n^2}{u^2} - \frac{1}{6} \frac{m^2}{n^2} + \left(T - \frac{1}{3} \frac{J_P'}{J} + \frac{1}{3} \frac{n}{u}\right) \frac{\sin\vartheta}{R_0} +$$

$$+ \left(-\frac{2}{9} T - \frac{1}{18} \frac{J_P'}{J} + \frac{1}{18} \frac{n}{u}\right) \frac{\cos\vartheta}{R_0} + \frac{5}{6} \frac{\sin^2\vartheta}{R_0^2} - \frac{1}{18} \frac{\cos^2\vartheta}{R_0^2} - \frac{1}{12} \frac{\sin 2\vartheta}{R_0^2}.$$

We obtain for the curvature k_φ and its derivative k'_φ at the starting point that

$$k_\varphi = \frac{1}{\sqrt{2}} \left[-\left(\frac{2}{3} T + \frac{1}{2} \frac{\sin\vartheta}{R_0}\right) + \frac{1}{3} \frac{J_P'}{J} - \frac{1}{3} \frac{n}{u} + \frac{1}{6} \frac{\cos\vartheta}{R_0} \right],$$

$$k_\varphi' = \frac{1}{2\sqrt{2}} \left[-2(\varkappa_1^2 + \varkappa_2^2 - \varkappa_1\varkappa_2) - \left(k_2 - \frac{n}{u}\right) \frac{J_P'}{J} + \frac{J_P''}{J} - \frac{J_P'^2}{J^2} + \frac{J}{u^3} + \right. \tag{4.9}$$

$$\left. + \frac{n^2}{u^2} - \frac{m^2}{u^2} + 2T \frac{\sin\vartheta}{R_0} + \left(-\frac{J_P'}{J} + \frac{n}{u}\right) \frac{\cos\vartheta}{R_0} + 3 \frac{\sin^2\vartheta}{R_0^2} - \frac{1}{2} \frac{\cos 2\vartheta}{R_0^2} \right].$$

We have exhausted all the possible emission modes. Notice that the above expressions refer to plane flows when $R_0 \to \infty$ and $\varkappa_2 = k_2 = 0$. The angles of 67.5° (ρ-mode), 90° ($\varepsilon \neq 0$), and 45° ($\varepsilon = 0$, $u \neq 0$), formed by the zero equipotential with the beam boundary, are characteristic and independent of the emitting surface geometry, the magnetic fields, or the current density and field distributions on the emitter. These parameters affect the subsequent terms in the equation of the zero equipotential (the curvature and its derivative, in cases in which these characteristics have any meaning).

REFERENCES

1. V. A. Syrovoi, "The curvature of the zero shaping electrode," PMTF [Journal of Applied Mechanics and Technical Physics], no. 5, 1967.
2. Yu. E. Kuznetsov and V. A. Syrovoi, "Solution of the equations of a regular electrostatic beam in the presence of emission from arbitrary surface," PMTF [Journal of Applied Mechanics and Technical Physics], no. 2, 1966.
3. V. A. Syrovoi, "Solution of regular beam equations in arbitrary emission conditions on a curvilinear surface," PMTF [Journal of Applied Mechanics and Technical Physics], no. 3, 1966.
4. V. A. Syrovoi, "Solution of the equations for a regular beam emitted by a curvilinear surface in the nonstationary case," PMTF [Journal of Applied Mechanics and Technical Physics], no. 6, 1966.
5. D. E. Radley and A. B. Birtles, "Approximate solution to the electron-flow equations," Internat. J. Electr., vol. 21, no. 5, 1966.
6. K. J. Harker, "Solution of the Cauchy problem for Laplace's equation in axially symmetric systems," J. Math. Phys., vol. 4, no. 7, 1963.

28 November 1967

Moscow